$\therefore v^2=gr$. $\therefore (v^2/g)\sin 2\theta = r\sin 2\theta$.

The radii of the three concentric circles are $\frac{1}{2}r\sqrt{3}$, $\frac{1}{2}r\sqrt{2}$, $\frac{1}{2}r$, respectively.

$\therefore r\sin 2\theta=\frac{1}{2}rn$, suppose. Solving, we get $\sin\theta=\frac{1}{2}\sqrt{[2\pm\sqrt{(4-n^2)}]}$.

When $n=\sqrt{3}$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm1)}$. $\therefore \theta=\frac{1}{3}\pi$ or $\frac{1}{6}\pi$.

When $n=\sqrt{2}$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm\sqrt{2})}$. $\therefore \theta=3\pi/8$ or $\frac{1}{8}\pi$.

When $n=1$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm\sqrt{3})}$. $\therefore \theta=5\pi/12$ or $\frac{1}{12}\pi$.

Chance that all fall into outer ring$=p$.

$$\therefore p=\frac{r\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\sin 2\theta d\theta}{r\int_0^{\frac{1}{2}\pi}\sin 2\theta d\theta}=\tfrac{1}{2}.\quad p_1=\frac{r\int_{\frac{1}{8}\pi}^{3\pi/8}\sin 2\theta d\theta}{r\int_0^{\frac{1}{2}\pi}\sin 2\theta d\theta}=\tfrac{1}{2}\sqrt{2},$$

the chance that all will fall in the two outer rings.

$\therefore P=p_1-p=\frac{1}{2}(\sqrt{2}-1)$=the chance that all will fall in the second ring from without.

$$p_2=\frac{r\int_{\frac{1}{12}\pi}^{5\pi/12}\sin 2\theta d\theta}{r\int_0^{\frac{1}{2}\pi}\sin 2\theta d\theta}=\tfrac{1}{2}\sqrt{3},$$ the chance that all will fall in three outer rings.

$\therefore P_1=p_2-p_1=\frac{1}{2}(\sqrt{3}-\sqrt{2})$=chance that all fall in third ring from without.

$\therefore P_2=1-p_2=\frac{1}{2}(2-\sqrt{3})$=chance that all fall in small circle around the center.

$\therefore$ Number to fall in each space is proportional to these chances, or as $1:(\sqrt{2}-1):(\sqrt{3}-\sqrt{2}):(2-\sqrt{3})$.

$\therefore$ Number in outer ring$=\frac{1}{2}m=500$.

Number in second ring$=m(\sqrt{2}-1)/2=500(\sqrt{2}-1)=207.1065$.

Number in third ring$=m(\sqrt{3}-\sqrt{2})/2=500(\sqrt{3}-\sqrt{2})=158.9185$.

Number in inside circle$=m(2-\sqrt{3})/2=500(2-\sqrt{3})=133.9750$.

93. **Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr. University, Palo Alto, Cal.**

In Problem 75, required the average area of the circle inscribed in the triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let r=radius of inscribed circle.

Then $\frac{1}{2}pr=\text{area}=\frac{1}{2}r[x+y+\sqrt{(x^2+y^2)}]=\frac{1}{2}xy$.

But $x=\dfrac{p^2-2py}{2(p-y)}$. $\therefore \frac{1}{2}pr=\dfrac{p(p-2y)y}{4(p-y)}$.

$$\therefore r=\frac{(p-2y)y}{2(p-y)},\quad \pi r^2=\frac{\pi y^2(p-2y)^2}{4(p-y)^2}.$$

The limits of y are 0 and $\frac{p}{2+\sqrt{2}}=y'$.

$$\therefore \Delta=\frac{\frac{1}{4}\pi\int_0^{y'}\frac{y^2(p-2y)^2}{(p-y)^2}dy}{\int_0^{y'}dy}=\frac{\pi(2+\sqrt{2})}{4p}\int_0^{y'}\left(4y^2+4py+5p^2+\frac{p^4}{(p-y)^2}-\frac{6p^3}{p-y}\right)dy$$

$$=\frac{\pi p^2}{12}[27-4\sqrt{2}-9(2+\sqrt{2})\log 2].$$

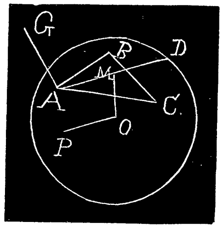
In this solution, as in solution of problem 75, I used the limits of y, 0 and $p/(2+\sqrt{2})$. These limits give all possible variations of size of area. Any other areas are mere repetitions of those included in the above and such a repetition or doubling of areas I believe to be inadmissible.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Three points are taken at random on the surface of the sphere. Find the chance that the triangle thus formed is acute angled.

Solution by the PROPOSER.

Let AD be the diameter of the section of the sphere made by the plane through the three random points A, B, C; M its center; O the center of the sphere; OP a line such that AB is parallel to the plane MOP; p=the chance.



Let $AO=r$, $\angle AOM=\theta$, $\angle GAC=\varphi$, $\angle GAB=\psi$, $\angle MOP=\lambda$, the angle MOP makes with some fixed plane through $OP=\rho$.

An element of the sphere at A is $4\pi r^2\sin\theta d\theta$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $\frac{1}{2}\pi$ and π; of ψ, $\pi-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π.

The three points can be taken $64\pi^3r^6$ ways on the surface of the sphere. Hence

$$p=\frac{1}{64\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{2\pi}\int_0^{\pi}\int_0^{2\pi}4\pi r^2\sin\theta d\theta.4r^2\sin\theta\sin\varphi d\varphi d\lambda$$

$$\times 4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)\sin\lambda d\theta d\varphi d\psi d\lambda$$

$$=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)d\theta d\varphi d\psi$$